JOURNAL OF APPROXIMATION THEORY 11, 216–220 (1974)

A Markov Inequality in Several Dimensions*

DON R. WILHELMSEN

Department of Mathematics, University of Georgia, Athens, Georgia 30602

Communicated by Philip J. Davis

1. INTRODUCTION

If p is a polynomial of degree k or less whose modulus is bounded by one on $[-1, 1]$, then

$$\max_{t\in[-1,1]} | p'(t)| \leqslant k^2. \tag{1.1}$$

This result was first proved by A. Markov [7] and later generalized for higher derivatives by W. Markov [8]. Equality occurs in (1.1) if and only if p is the kth Tchebycheff polynomial of the first kind. Duffin and Schaeffer [1] demonstrated a more fundamental connection between (1.1) and the Tchebycheff polynomials: One need only assume $| p(\cos[\nu\pi/k])| \leqslant 1$, $\nu = 0, 1,..., k$, in order for the same conclusion to hold. Modifications of (1.1) have been studied by Hille, *et al.* [3], Scheick [9], and others. Observe that if $| p(t)| \leqslant 1$ for $t \in [-d, d]$, then

$$| p'(t)| \leqslant \frac{k^2}{d}, \qquad t \in [-d, d]. \tag{1.2}$$

Inequality (1.2) is essential in the study of oscillatory properties of polynomials. During the author's investigation of numerical integration in several variables [11], a multidimensional analog was required; it was given for arbitrary convex, compact sets in Euclidean n-space. That result is improved in this paper, using a considerably simplified approach.

O. D. Kellogg [6] obtained an analog of (1.2) for the sphere in E_n. His bound is sharp, and to the author's knowledge this is the only other extension of the Markov inequality to several dimensions.

* This work formed a part of the author's Ph.D. thesis completed at Brown University, Division of Applied Mathematics, and supported by an NDEA Title IV Fellowship.

2. Preliminaries

Throughout this paper T will be a compact, convex set in E_n with boundary ∂T and nonempty interior T^0. The Euclidean norm of an n-vector v will be denoted by $\| v \|$, and we shall define set-related norms for a continuously differentiable function p and its gradient ∇p by

$$\| p \|_T \equiv \max_{t \in T} | p(t)| \tag{2.1}$$

and

$$\| \nabla p \|_T \equiv \max_{t \in T} \| \nabla p(t)\|. \tag{2.2}$$

Let $\mathscr{P}_{k,T}$ be the set of algebraic polynomials of total degree k or less in n variables which satisfy the condition

$$\| p \|_T \leqslant 1, \qquad p \in \mathscr{P}_{k,T}. \tag{2.3}$$

For $k = 1, 2,...,$ define

$$M_{k,T} \equiv \max\{\| \nabla p \|_T , p \in \mathscr{P}_{k,T}\},$$

which we shall refer to as Markov numbers. Our aim is to find upper bounds for the Markov numbers. (For a general exposition of multivariate polynomials, the reader is referred to Hirsch [4], Stroud [10], and Jackson [5].)

Fix $t_0 \in \partial T$, let u be a unit vector, and consider the hyperplane with normal u (see Goldstein [2])

$$\mathscr{H}_u \equiv \{t \in E_n: (t - t_0 , u) = 0\},$$

where $(\cdot, \cdot)$ denotes scalar product. $\mathscr{H}_u$ is a support hyperplane of T at t_0 if and only if it contains no interior points of T. One may arrange things so that $(t, u) \leqslant 0$ when $t \in T$, in which case u is called an outer normal to T at t_0. The following facts are easily verified and will be used later (recall that T is convex).

(i) If $t_0 \in \partial T$, there exists at least one support hyperplane for T at t_0.

(ii) For any direction u, there exist precisely two support hyperplanes of T, one with outer normal u and the other with outer normal $-u$. They are separated by a distance $\lambda_u > 0$.

(iii) If $\mathscr{H}_u$ is not a support hyperplane, then one can find two points t_1 and t_2 in T such that $(t_1 - t_0 , u) < 0$ and $(t_2 - t_0 , u) > 0$, where $t_0 \in \mathscr{H}_u \cap \partial T$.

(iv) If the distance of t from $\mathscr{H}_u$ is d and $t_0 \in \mathscr{H}_u$, then $|(t - t_0 , u)| = d$.

DEFINITION 2.1. Define the width of T to be

$$\omega_T \equiv \min_{\|u\|=1} \lambda_u .$$

Note that $\omega_T > 0$, given our assumptions on T.

Finally, we shall use the two properties of polynomials listed below.

(v) Fix $t_0 \in E_n$. If $p(t)$ is of degree k or less in n variables, then $p(t_0 + \lambda u)$ is of degree k or less in λ.

(vi) Let u be a unit vector. The derivative of p at t_0 in the direction u is given by

$$p_u(t_0) \equiv \frac{d}{d\lambda} p(t_0 + \lambda u)\,|_{\lambda=0} = (\nabla p(t_0), u).$$

3. DERIVATION OF BOUNDS

LEMMA 3.1. *Given* $p \in \mathscr{P}_{k,T}$, *let* $t^* \in T$ *satisfy* $\| \nabla p(t^*)\| = \| \nabla p \|_T$. *If also* $| p(t^*)| = 1$, *then*

$$\| \nabla p \|_T \leqslant \frac{2k^2}{\omega_T}. \tag{3.1}$$

Proof. We omit the trivial case when p is constant. Since $| p(t)|$ cannot exceed 1, we must have $t^* \in \partial T$. Let $u = \nabla p(t^*)/\| \nabla p(t^*)\|$, and let $\mathscr{H}_u$ be the hyperplane with normal u which passes through t^*. We claim $\mathscr{H}_u$ is a support hyperplane of T. If not, and if $p(t^*) = 1$, we can find a point $t_1 \in T$ for which by (vi) $p_v(t^*) = (\nabla p(t^*), v) > 0$, where $v = (t_1 - t^*)/\| t_1 - t^* \|$. Since this implies $| p(t)|$ must exceed 1 somewhere on the segment $[t^*, t_1]$, we have a contradiction. The case $p(t^*) = -1$ follows similarly.

Now, assume u is an outer normal for T (or arrange things so), and let $\mathscr{H}_{-u}$ be the support hyperplane with outer normal $-u$ lying a distance λ_u from $\mathscr{H}_u$. Let $t_0 \in \mathscr{H}_{-u} \cap T$. The line segment $[t_0, t^*]$ lies in T, and $p(t^* + \lambda w)$, where $w = (t_0 - t^*)/\| t_0 - t^* \|$, is a polynomial in λ of degree k or less which is bounded by 1 on $[0, \| t_0 - t^* \|]$. Since polynomials are translation invariant, inequality (1.2) yields

$$\begin{aligned}\frac{2k^2}{\| t_0 - t^* \|} \geqslant | p_w(t^*)| &= |(\nabla p(t^*), w |\\ &= \frac{\lambda_u \| \nabla p(t^*)\|}{\| t_0 - t^* \|}\\ &\geqslant \frac{\omega_T \| \nabla p(t^*)\|}{\| t_0 - t^* \|},\end{aligned}$$

from which (3.1) follows.

THEOREM 3.1. *If* $p \in \mathscr{P}_{k,T}$, *then*

$$\| \nabla p \|_T < \frac{4k^2}{\omega_T}. \tag{3.2}$$

Proof. Find $t^* \in T$ such that $\| \nabla p(t^*)\| = \| \nabla p \|_T$ and assume p is not constant. If $| p(t^*)| = 1$, we cite the previous Lemma. Otherwise, let $u = \nabla p(t^*)/\| \nabla p(t^*)\|$. Let $\mathscr{H}_u$ and $\mathscr{H}_{-u}$ be the support hyperplanes of T with outer normals u and $-u$, respectively. Let $t_0 \in \mathscr{H}_u \cap T$ and $t_1 \in \mathscr{H}_{-u} \cap T$. Clearly t^* must be a distance $(\lambda_u/2) \geqslant (\omega_T/2)$ or more from $\mathscr{H}_u$ or $\mathscr{H}_{-u}$. Suppose it is $\mathscr{H}_u$. Since $| p(t^*)| < 1$, we may travel a little distance from t^* in the direction $(t^* - t_0)$ to a point t_δ such that

(a) $| p(t)| \leqslant 1$ on $[t_0, t_\delta]$ and
(b) the distance of t_δ from $\mathscr{H}_u$ is $\lambda_\delta > \omega_T/2$.

We now repeat the argument of Lemma 3.1 to get

$$\begin{aligned} \frac{2k^2}{\| t_\delta - t_0 \|} &\geqslant \left| \left(\nabla p(t^*), \frac{t_\delta - t_0}{\| t_\delta - t_0 \|} \right) \right| \\ &= \frac{\lambda_\delta \| \nabla p(t^*)\|}{\| t_\delta - t_0 \|} \\ &> \frac{\omega_T \| \nabla p(t^*)\|}{2 \| t_\delta - t_0 \|}. \end{aligned}$$

THEOREM 3.2.

$$M_k < \frac{4k^2}{\omega_T}, \qquad k = 1, 2, \ldots. \tag{3.3}$$

Proof. If $k \geqslant 1$, we note that $\| \nabla p \|_T$ is a continuous function on the compact set $\mathscr{P}_{k,T}$. The rest follows from Theorem 3.1.

4. A SHARPNESS CONJECTURE

For the unit ball $B \subset E_n$ Kellogg [6] showed that $M_k \leqslant k^2$. This is made sharp by the extremal polynomials $\cos[k \cos^{-1} \lambda(t)]$, where $\lambda(t)$ is the signed distance of t from a fixed hyperplane passing through the center of B. Note that $\omega_B = 2$, so that the bound of Theorem 3.2 differs from M_k by a factor less than 2.

We conjecture that for arbitrary convex, compact T, $M_k = 2k^2/\omega_T$, $k = 0, 1, 2, \ldots$. This is equivalent to proposing that there always exists an extremal polynomial p and a point $t^* \in T$ such that $M_k = \| \nabla p(t^*)\|$ and $| p(t^*)| = 1$.

A stronger conjecture would be the following: Any extremal polynomial must attain its maximum derivative value and maximum magnitude coincidentally at some point in ∂T. This property is satisfied by the Tchebycheff polynomials mentioned above. It would be interesting to determine whether or not these are the only extremal polynomials.

Acknowledgment

I am indebted to Professor Philip J. Davis for bringing to my attention the reference by Kellogg.

References

1. R. J. Duffin and A. C. Schaeffer, A refinement of an inequality of the brothers Markoff, *Trans. Am. Math. Soc.* **50** (1941), 517–528.
2. A. A. Goldstein, "Constructive Real Analysis," Harper and Row, New York, 1967.
3. E. Hille, G. Szegö and J. D. Tamarkin, On some generalizations of a theorem of A. Markoff, *Duke Math. J.* **3** (1937), 729–739.
4. P. M. Hirsch, Evaluation of orthogonal polynomials and its relationship to evaluating multiple integrals, *Math. Comp.* **22** (1968), 280–285.
5. D. Jackson, Formal properties of orthogonal polynomials in two variables, *Duke Math. J.* **2** (1936), 423–434.
6. O. D. Kellogg, On bounded polynomials in several variables, *Math. Zeit.* **27** (1927), 55–64.
7. A. Markov, Sur une question posee par Mendeleieff, *Bull. Acad. Sci. St. Petersburg* **62** (1889), 1–24.
8. W. Markov, Uber polynome, die in einem gegebenen intervalle moglichst wenig von null abweichen, *Mat. Ann.* **77** (1916), 213–258.
9. J. T. Scheick, Inequalities for derivatives of polynomials of special type, *J. Approx. Theor.* **6** (1972), 354–358.
10. A. H. Stroud, Integration formulas and orthogonal polynomials for two variables, *SIAM J. Numer. Anal.* **6** (1969), 222–229.
11. D. R. Wilhelmsen, Non-negative approximate integration in several dimensions, Ph.D. thesis, Brown University, 1972.